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Oscillations of a vapour cavity in a rotating cylindrical tank

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The oscillations of a curved interface are considered, neglecting the effects of gravity. The system under consideration consists of a right, circular, cylindrical tank of finite length, partially filled with an inviscid, incompressible, wetting liquid. When the container spins about its axis of revolution, the large-scale vapour cavity takes an elongated spheroid-like shape, symmetric about the axis of rotation. The fluid–vapour interface will oscillate about the equilibrium configuration if disturbing forces are present. The case where the vapour cavity touches the walls of the tank is not included in this investigation.

The equations of motion are linearized. However, the resulting eigenvalue problem is non-linear. Surface tension and rotation are taken into account only to the extent allowed by a linearized stability theory.

The self-sustained oscillations are governed by a partial differential equation of elliptic type, the field equation of the perturbation pressure. According to the results obtained from theory, all eigenfrequencies for this case are greater than twice the angular speed of the tank. The first two eigenfrequencies can be computed with high accuracy. The relation between the bubble shape and the eigenfrequency is shown in a graph for a specific example.

The governing differential equation is hyperbolic for forced oscillations induced by a small force field of constant magnitude and direction in an inertial frame of reference. A solution for this problem exists only in case of a cylindrical tank of infinite length. Discontinuities in the velocity components occur in the flow field. A numerical example has been carried out.

1. Introduction

In low gravitational environments, forces like surface tension and the centrifugal force, induced by slow rotation of the fluid, will have a dominating effect on the large-scale equilibrium configuration and the dynamics of a fluid system. For example, a right circular cylindrical tank, partially filled with a wetting liquid, spins about its axis of revolution with constant angular speed and is placed in a weak gravitational (zero-g) field. Then the surface tension at the interface, together with the centrifugal force, cause the vapour cavity (bubble) inside the

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vessel to take an elongated spheroid-like shape, situated axially symmetrically about the axis of rotation. Equilibrium configurations for various constant angular speeds have been studied by Rosenthal (1962). A concise survey on the literature related to this field is given in a review article by Habip (1965).

The present paper deals with the exploration of the nature of the liquid interface oscillations in a rotating tank and the possibility of rotational stabilization.

The chosen frame of reference is fixed in the tank and rotates with it. The perturbation velocities, the interface wave amplitude, and the disturbing forces are assumed to be small. Thus in the equations of motion, the equation of continuity, and the boundary conditions terms involving quadratic or higher orders of the perturbation quantities are neglected.

The fluid is assumed to be inviscid and incompressible. Surface tension and rotational velocity components are both essential in the study. They are taken fully into account as far as a linearized theory will allow.

For this rotating fluid oscillation problem there are two frequency ranges for which the disturbances are of an entirely different character. For $\omega > 2$, the type of the governing partial differential equation is elliptic, while for $\omega < 2$, the flow field is described by a hyperbolic equation. The method of analysis and the physical interpretation of the flow phenomena are entirely different.

For the elliptic case, the bubble shape is approximated by a prolate spheroid. A spheroidal co-ordinate system is found useful because it allows a straightforward analysis. Hence, associated Legendre functions can be employed for the series expansion of the solution. A special method has been employed in order to account for the homogeneous boundary condition at the wall of the tank. The resulting eigenvalue problem for the relative frequency of oscillation is nonlinear, since the eigenfrequency is included as a parameter in the formulation of the differential equation as well as the boundary conditions. The eigenfrequencies are obtained through a successive approximation procedure. This procedure proved to be rapidly convergent for the numerical computations performed for a sequence of examples.

The result for this case indicates, that for the circumferential mode $m = 2$, the first eigenfrequency greater than two corresponds to the first mode of vibration in the meridian plane. Hence, for higher modes of oscillation the flow field is certainly elliptic in nature.

For a forced oscillation, induced by a reduced gravity field of constant magnitude and direction in an inertial frame of reference, the relative frequency of oscillation is less than two. The governing differential equation is hyperbolic, hence the method of analysis is different. Consequently, the structure of the flow field is different from that of the above-mentioned elliptic case. The mathematical problem (Cauchy and Goursat problem) is transformed into integro-differential equations which can be integrated numerically by means of Picard's method of successive approximations. It is found that a steady solution in a cylindrical tank of finite length does not exist. However, a solution exists in a cylindrical tank of infinite length. The perturbations extend to infinity. A numerical example for this forced oscillation problem has been carried out.

In the present work we consider only oscillations which are symmetric with respect to the equatorial plane of the bubble, i.e. symmetric with respect to the variable ζ in the cylindrical co-ordinate system, and with respect to the variable β in the prolate spheroidal co-ordinate system.

2. Field equation for the perturbation pressure

The equations of motion governing the fluid flow in a rotating frame of reference can be written in the following form (Morgan 1951):

$$\frac{\partial \mathbf{q}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{q} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \mathbf{q} . \nabla \mathbf{q} = \mathbf{F} - \frac{1}{\rho} \nabla \hat{p},$$

where $\mathbf{q}$ denotes the relative (perturbation) velocity vector, $\mathbf{\Omega}$ the constant angular velocity vector, $\mathbf{r}$ the position vector from the origin of the rotating coordinate system to the point occupied by the fluid particle, and $\mathbf{F}$ the external force per unit mass of the fluid. We further denote the density of the liquid by ρ and the pressure by $\hat{p}$. In the present problem the basic fluid flow is in a state of rigid body rotation with respect to the inertial frame of reference and hence is at rest with respect to the rotating, non-translating frame as referred to in this investigation. By neglecting the quadratic terms of the perturbation velocity components a linearized system of equations is obtained. This system can be written in a cylindrical co-ordinate system (r, θ, z) as

$$\left.\begin{aligned} \frac{\partial \hat{u}}{\partial t} - 2\Omega \hat{v} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial r} + F_r, \\ \frac{\partial \hat{v}}{\partial t} + 2\Omega \hat{u} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \theta} + F_\theta, \\ \frac{\partial \hat{w}}{\partial t} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial z} + F_z, \end{aligned}\right\} \tag{1}$$

with $\{\hat{u}, \hat{v}, \hat{w}\}$ as the components of the perturbation velocity vector, $\{F_r, F_\theta, F_z\}$ as the components of the external force, and Ω as the constant angular speed of the rotating tank.

The equation of continuity reads

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}) + \frac{1}{r}\frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial z} = 0. \tag{2}$$

The system of differential equations can be made dimensionless by choosing the semi-minor axis c of the undisturbed bubble as the length scale and Ω^{-1} as the time scale. Further, in our case, the acting external perturbing force is assumed to originate from a force field which is transverse to the direction of the tank axis. This field has a constant direction and magnitude in the inertial frame of reference and hence will appear to be oscillatory in the chosen rotating frame of reference. Then the pressure can be written as

$$\hat{p} = \tfrac{1}{2}\rho\Omega^2 c^2\{\eta^2 + 2p\} + \epsilon\rho c\eta\, e^{i(\Omega t + \theta)} + \hat{p}_{0L}, \tag{3}$$

where p is the dimensionless perturbation pressure, $\hat{p}_{0L}$ is a constant, and the time-dependent, second term on the right-hand side takes account of the perturbation force effects. The constant acceleration transverse to the axis of rotation is denoted by ϵ. Its direction is fixed in the inertial frame of reference. The dimensionless equations of motion and the equation of continuity are given as:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2v &= -\frac{\partial p}{\partial \eta}, \\ \frac{\partial v}{\partial t} + 2u &= -\frac{1}{\eta}\frac{\partial p}{\partial \theta}, \\ \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial \zeta}, \\ \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta u) + \frac{1}{\eta}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \zeta} &= 0, \end{aligned}\right\} \tag{4}$$

where $\eta = r/c$, $\zeta = z/c$ and $\{u, v, w\}$ are the dimensionless perturbation velocity components, respectively.

We further define the following dimensionless numbers

$$E = \rho\Omega^2 c^3/8T, \quad B = \epsilon\rho c^2/T,$$

where E is the ratio of the magnitude of the centrifugal force to the surface tension force and B is the ratio of the magnitude of the transverse perturbation force field to the surface tension force. The coefficient of surface tension is denoted by T. For a small disturbance, B is required to be much smaller than E. By eliminating the velocity components from the continuity equation, we obtain the governing field equation for the fluid flow

$$\frac{\partial^2}{\partial t^2}\left\{\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial p}{\partial \eta}\right) + \frac{1}{\eta^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial \zeta^2}\right\} + 4\frac{\partial^2 p}{\partial \zeta^2} = 0. \tag{5}$$

This partial differential equation exhibits some very distinct properties. Let us assume that the perturbation pressure can be written in the form

$$p(\eta, \theta, \zeta, t) = P(\eta, \theta, \zeta)\, e^{i\omega t}, \tag{6}$$

where ω denotes the perturbation frequency relative to the rotating system, i denotes the imaginary unit and the real part of the right-hand side has to be used for physical interpretation.

With this set-up the field equation for the perturbation pressure is transformed into

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial P}{\partial \eta}\right) + \frac{1}{\eta^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{\omega^2 - 4}{\omega^2}\frac{\partial^2 P}{\partial \zeta^2} = 0, \tag{7}$$

and it follows that the foregoing equation is of elliptic type when $\omega > 2$, and of hyperbolic type when $\omega < 2$. The last term on the left-hand side of (7) vanishes if $\omega = 2$. Similarly, the velocity components can then be expressed explicitly in terms of the perturbation pressure and its derivatives:

$$(\omega^2 - 4)\, u = \left[i\omega\frac{\partial P}{\partial \eta} + \frac{2}{\eta}\frac{\partial P}{\partial \theta}\right] e^{i\omega t},$$

$$(\omega^2 - 4)\, v = \left[i\omega\frac{1}{\eta}\frac{\partial P}{\partial \theta} - 2\frac{\partial P}{\partial \eta}\right] e^{i\omega t},$$

$$\omega^2 w = i\omega\frac{\partial P}{\partial \zeta} e^{i\omega t}.$$

Now let the perturbation velocity $\{u, v, w\}$ be divided into two parts $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$, such that

$$u_1 = \frac{i}{\omega}\frac{\partial P}{\partial \eta} e^{i\omega t}, \quad u_2 = \left[\frac{2}{\omega^2-4}\frac{1}{\eta}\frac{\partial P}{\partial \theta} + \frac{4i}{\omega(\omega^2-4)}\frac{\partial P}{\partial \eta}\right] e^{i\omega t},$$

$$v_1 = \frac{i}{\omega}\frac{1}{\eta}\frac{\partial P}{\partial \theta} e^{i\omega t}, \quad v_2 = \left[\frac{-2}{\omega^2-4}\frac{\partial P}{\partial \eta} + \frac{4i}{\omega(\omega^2-4)}\frac{1}{\eta}\frac{\partial P}{\partial \theta}\right] e^{i\omega t},$$

$$w_1 = \frac{i}{\omega}\frac{\partial P}{\partial \zeta} e^{i\omega t}, \quad w_2 = 0.$$

The velocity component $\{u_1, v_1, w_1\}$ is irrotational because it is the gradient of a scalar function. For the velocity component $\{u_2, v_2, w_2\}$, we have

$$\operatorname{curl} \mathbf{v} = \operatorname{curl} \mathbf{v}_2 = \left\{\left[\frac{-4i}{\omega(\omega^2-4)}\frac{1}{\eta}\frac{\partial^2 P}{\partial \theta\, \partial \zeta} + \frac{2}{\omega^2-4}\frac{\partial^2 P}{\partial \eta\, \partial \zeta}\right] e^{i\omega t}, \right.$$
$$\left. \left[\frac{4i}{\omega(\omega^2-4)}\frac{\partial^2 P}{\partial \eta\, \partial \zeta} + \frac{2}{\omega^2-4}\frac{1}{\eta}\frac{\partial^2 P}{\partial \theta\, \partial \zeta}\right] e^{i\omega t}, \quad \left[\frac{2}{\omega^2}\frac{\partial^2 P}{\partial \zeta^2}\right] e^{i\omega t}\right\},$$

where $$\mathbf{v} = \{u, v, w\} \quad \text{and} \quad \mathbf{v}_2 = \{u_2, v_2, w_2\}.$$

Hence, the velocity field can be characterized as follows.

(*a*) For $\omega > 2$, the magnitude of the rotational velocity component is smaller than the magnitude of the irrotational one. Their ratio tends to zero at the rate $1/\omega$ as ω tends to infinity.

(*b*) For $\omega < 2$, the magnitude of the rotational velocity component is larger than the magnitude of the irrotational one. Their ratio tends to infinity of the order $O(1/\omega^2)$ as ω tends to zero. The vorticity effect is so dominating that the velocity field may have a cellular structure. An example can be found in Phillips (1960).

(*c*) For $\omega = 2$, the partial differential equation is indeterminate in the sense that the ζ-dependence is arbitrary. Any solution of the form

$$p(\eta, \theta, \zeta, t) = \{\kappa(\eta, \theta)\, \varpi_1(\zeta) + \varpi_2(\zeta)\}\, e^{i\omega t},$$

where $\kappa(\eta, \theta)$ satisfies the field equation and the boundary conditions and any functions $\varpi_1(\zeta)$ and $\varpi_2(\zeta)$, which fulfil the boundary conditions, are admissible. The solution is not unique. Hence, in the linearized theory such a flow is unstable.

If the tank–liquid system rotates at a constant angular speed, the equilibrium bubble interface is a closed surface of revolution about the axis of rotation. The exact shape of the bubble has been obtained explicitly by various investigators. The bubble is elongated along the axis. As indicated in Rosenthal (1962), the bubble is similar to, and slightly larger than the prolate spheroid with the same semi-minor and semi-major axis. For the case of no rotation, the shape is a sphere and in the limiting case the bubble elongates to a tube of infinite length when $E = 0{\cdot}5$.

In the following sections the problem is solved separately for the frequency ranges $\omega > 2$ and $\omega < 2$.

3. Solution for the elliptic case, $\omega > 2$

In order to allow a straightforward analysis, the exact bubble shape is approximated by a prolate spheroid which has the same semi-major and semi-minor axes as the exact bubble. With this approximation, the perturbation pressure field can be expanded in terms of associated Legendre functions, and the boundary conditions can be formulated also at the approximate location.

The affine transformation of the space co-ordinates

$$\eta = \eta, \quad \theta = \theta, \quad \mu = \frac{\omega}{(\omega^2-4)^{\frac{1}{2}}}\zeta, \tag{8}$$

will reduce the field equation (7) to

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial P}{\partial\eta}\right)+\frac{1}{\eta^2}\frac{\partial^2 P}{\partial\theta^2}+\frac{\partial^2 P}{\partial\mu^2} = 0. \tag{9}$$

Another co-ordinate transformation is needed for the analysis of this problem. The cylindrical co-ordinates (η,θ,μ) are transformed into prolate spheroidal co-ordinates (α,β,θ) by means of

$$\eta = \tfrac{1}{2}a[(\alpha^2-1)(1-\beta^2)]^{\frac{1}{2}}, \quad \mu = \tfrac{1}{2}a\alpha\beta, \quad \theta = \theta, \tag{10}$$

where a is a scale constant which will be determined by the shape of the bubble.

In this co-ordinate system the equation for the pressure field becomes

$$\frac{1}{\alpha^2-\beta^2}\left\{\frac{\partial}{\partial\alpha}\left[(\alpha^2-1)\frac{\partial P}{\partial\alpha}\right]+\frac{\partial}{\partial\beta}\left[(1-\beta^2)\frac{\partial P}{\partial\beta}\right]+\left[\frac{1}{\alpha^2-1}+\frac{1}{1-\beta^2}\right]\frac{\partial^2 P}{\partial\theta^2}\right\} = 0. \tag{11}$$

It should be noted that the field equation is separable in the cylindrical co-ordinates (η,θ,μ) as well as in the prolate spheroidal co-ordinates (α,β,θ).

The next step in our analysis is to formulate the boundary conditions at the rigid tank wall and the free liquid-vapour interface.

At the plane end disks of the tank the normal velocity, which is in the μ-direction, must vanish. Thus

$$\frac{\partial P}{\partial\mu} = 0 \quad \text{at} \quad \mu = \frac{\pm\omega l}{(\omega^2-4)^{\frac{1}{2}}}, \tag{12}$$

where $l = L/2\,c$ with L as the length of the tank. At the cylindrical portion of the tank wall the normal velocity component, which is in the η-direction, must vanish. Hence,

$$i\omega\frac{\partial P}{\partial\eta}+\frac{2}{\eta}\frac{\partial P}{\partial\theta} = 0 \quad \text{at} \quad \eta = \eta_0, \tag{13}$$

where $\eta_0 = R/c$ with R as the radius of the tank. The bubble interface shall first be specified in the spheroidal co-ordinate system before the boundary conditions are set up. By taking into account the stretching due to the affine co-ordinate transformation, the bubble can be identified as one of the co-ordinate surfaces $\alpha = \alpha_0 =$ constant, where α_0 is given by

$$\frac{\alpha_0}{(\alpha_0^2-1)^{\frac{1}{2}}} = K\frac{\omega}{(\omega^2-4)^{\frac{1}{2}}}, \tag{14}$$

with K as the ratio of the semi-major axis to the semi-minor axis of the tank.

The scale constant a which appears in the relations (10), defining the coordinate transformation between (η, θ, μ) and (α, β, θ), is then

$$a = \frac{2}{(\alpha_0^2 - 1)^{\frac{1}{2}}}. \tag{15}$$

It is important to note here that for each different combination of bubble shape and frequency ω, the transformation is different. Hence the perturbation of the interface can be written in the form

$$\alpha - \alpha_0 = \alpha_1(\beta, \theta)\, e^{i\omega t}, \tag{16}$$

where the function $\alpha_1(\beta, \theta)$ is small such that $|\alpha_1| \ll (\alpha_0^2 - 1)$ and $|\alpha_1| \ll \alpha_0$.

Now a bounding surface $F(\alpha, \beta, \theta, t) = 0$ of a continuous medium is a material surface. Mathematically this condition can be written as

$$\frac{d}{dt}\{F(\alpha, \beta, \theta, t)\} = 0, \tag{17}$$

with
$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \alpha}\frac{d\alpha}{dt} + \frac{\partial F}{\partial \beta}\frac{d\beta}{dt} + \frac{\partial F}{\partial \theta}\frac{d\theta}{dt}. \tag{18}$$

In the present case we have

$$F(\alpha, \beta, \theta, t) \equiv \alpha - \alpha_0 - \alpha_1(\beta, \theta)\, e^{i\omega t} = 0, \tag{19}$$

hence
$$\frac{dF}{dt} = \frac{d\alpha}{dt} - i\omega\alpha_1(\beta, \theta)\, e^{i\omega t} - \frac{\partial \alpha_1}{\partial \theta}\frac{d\theta}{dt} e^{i\omega t} - \frac{\partial \alpha_1}{\partial \beta}\frac{d\beta}{dt} e^{i\omega t} = 0. \tag{20}$$

Neglecting quadratic terms in the above equation there results

$$\frac{d\alpha}{dt} - i\omega\alpha_1(\beta, \theta)\, e^{i\omega t} = 0. \tag{21}$$

The perturbation velocity component in the α-direction, v_α, may be written in terms of the pressure and its derivatives. It turns out that

$$v_\alpha = h_\alpha \frac{d\alpha_1}{dt}, \quad h_\alpha = \frac{a}{2}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - 1}\right)^{\frac{1}{2}}, \quad (\omega^2 - 4)\frac{d\alpha_1}{dt} = \frac{4i\omega}{a^2}\frac{(\alpha^2 - 1)}{(\alpha^2 - \beta^2)}\frac{\partial P}{\partial \alpha} + \frac{8\alpha}{a^2(\alpha^2 - \beta^2)}\frac{\partial P}{\partial \theta}, \tag{22}$$

where h_α denotes a scale factor. Then the above results can be summarized in the following equation

$$(\omega^2 - 4)\,\alpha_1(\beta, \theta)\, e^{i\omega t} = \frac{4(\alpha^2 - 1)}{a^2(\alpha^2 - \beta^2)}\frac{\partial P}{\partial \alpha} - \frac{i}{\omega}\frac{4}{a^2}\frac{2\alpha}{\alpha^2 - \beta^2}\frac{\partial P}{\partial \theta}. \tag{23}$$

A dynamical boundary condition shall also be satisfied besides the above kinematical boundary condition. The pressure at both sides of the interface, together with the surface tension force, must be in equilibrium with the centrifugal and inertia forces. Thus

$$\tfrac{1}{2}\rho\Omega^2 c^2(\eta^2 + 2p) + \epsilon\rho c\eta\, e^{i(\Omega t + \theta)}\big|_{\alpha = \alpha_0 + \alpha_1} + (\hat{p}_{0L} - \hat{p}_{0G}) = -(T/c)\, J^*, \tag{24}$$

where $\hat{p}_{0G}$ denotes the constant cavity pressure and J^* the curvature of the perturbed bubble.

Substituting now the proper quantities into the foregoing equation and subtracting the quantities which are originally in equilibrium (static equilibrium configuration of t herotating tank–fluid system with no disturbance forces acting) from both sides, we find

$$\rho\Omega^2c^2\{\tfrac{1}{4}a^2\alpha_0\alpha_1(1-\beta^2)+p_0\}+\epsilon\rho c\eta\, e^{i(\Omega t+\theta)} = -(1/c)\,T(J^*-J), \tag{25}$$

with p_0 as the perturbation pressure to be evaluated at the interface, and J^*-J as the change of curvature, where J denotes the curvature of the undisturbed bubble. Finally, this dynamic boundary condition can be put into the form

$$8E\{\tfrac{1}{4}a^2\alpha_0\alpha_1(1-\beta^2)+p_0\}+B\eta\, e^{i(\Omega t+\theta)} = -(J^*-J). \tag{26}$$

The change of curvature J^*-J is a lengthy expression in terms of β, θ, $\alpha_1(\beta,\theta)$, and the first and second derivatives of $\alpha_1(\beta,\theta)$ with respect to β. The procedure of calculation follows the method outlined in Struik (1950).

All the energy transfer and other important dynamical effects are contained in the boundary conditions at the free interface. The eigenfrequencies are determined by this boundary condition.

Now equation (11) in the prolate spheroidal co-ordinates is separable and after a few steps of standard manipulations the solution of the field equation for the perturbation pressure is obtained in the form:

$$p(\alpha,\beta,\theta) = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}[A_{mn}P_n^m(\alpha)+B_{mn}Q_n^m(\alpha)]\,P_n^m(\beta)\,e^{im\theta}. \tag{27}$$

In the above representation, the functions $P_n^m(\alpha)$, $P_n^m(\beta)$ and $Q_n^m(\alpha)$ are the Legendre associated functions of the first and the second kind, respectively. The coefficients A_{mn} and B_{mn} are constants to be determined by the boundary conditions. The dependence of the variable θ is given by $e^{im\theta}$, where $m = 0, 1, 2, \ldots$. The above eigenfunctions have been chosen to satisfy the boundary condition at the bubble interface. A suitable linear combination of them must now be chosen so that the condition at the outer rigid boundary is satisfied also. Circumferential modes for different values of m are linearly independent, the summation over m may be dropped and each mode handled separately.

In the following we shall construct a sequence of eigenfunctions which has the following properties: (*a*) the eigenfunctions are harmonic inside the tank; (*b*) each eigenfunction satisfies part of the required boundary conditions at the rigid wall of the tank; (*c*) any normal velocity perturbation at the wall of the tank can be expanded in a converging series in terms of the eigenfunctions in this sequence.

With this sequence of eigenfunctions, we can reduce the exterior boundary condition to the requirement that the surface integrals, formed between P and the first N eigenfunctions over the entire surface of the exterior boundary, are all zero

$$\oint_{\Sigma} P(\alpha,\beta,\theta)\,\Phi_n(\eta,\theta,\mu)\,d\sigma = 0 \quad (n = 1, 2, \ldots, N), \tag{28}$$

where the sequence of eigenfunctions is denoted by Φ_n and the entire surface of the exterior boundary is denoted by Σ.

Let P be represented in the cylindrical co-ordinate system by

$$P(\eta, \theta, \mu) = U(\eta)\, V(\theta)\, W(\mu).$$

Together with equation (9), we obtain the following system of uncoupled ordinary differential equations:

$$\left.\begin{aligned} \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dU}{d\eta}\right)-\left(\lambda^2+\frac{m^2}{\eta^2}\right)U &= 0,\\ \frac{d^2V}{d\theta^2}+m^2V &= 0,\\ \frac{d^2W}{d\mu^2}+\lambda^2W &= 0. \end{aligned}\right\} \tag{29}$$

We shall treat the cases $\lambda^2 > 0$ and $\lambda^2 < 0$ separately.

(*A*) *The case* $\lambda^2 > 0$

The boundary condition prescribed for this case is that the normal velocity component vanishes at the plane end disks:

$$(\omega^2-4)w = i\omega\frac{\partial P}{\partial \mu} = 0 \quad \text{at} \quad \mu = l^* = \frac{\omega l}{(\omega^2-4)^{\frac{1}{2}}}. \tag{30}$$

By means of equation (29) and this boundary condition, we obtain the following sequence of eigenfunctions

$$\phi_n = I_m\left(\frac{n\pi}{l^*}\eta\right)\cos\left(\frac{n\pi}{l^*}\mu\right)e^{i(\omega t+m\theta)}, \tag{31}$$

where I_m is the modified Bessel function of the first kind. The corresponding normal velocity component at the cylindrical wall is given as

$$u_n = \frac{in\pi}{(\omega^2-4)\,l^*}\left[\frac{\omega}{2}\left\{I_{m-1}\left(\frac{n\pi}{l^*}\eta_0\right)+I_{m+1}\left(\frac{n\pi}{l^*}\eta_0\right)\right\}\right.$$
$$\left.+\left\{I_{m-1}\left(\frac{n\pi}{l^*}\eta_0\right)-I_{m+1}\left(\frac{n\pi}{l^*}\eta_0\right)\right\}\right]\cos\left(\frac{n\pi}{l^*}\mu\right)e^{i(\omega t+m\theta)}. \tag{32}$$

The properties for this sequence of eigenfunctions can be summarized as follows: (*a*) the normal velocity component at the disk ends of the tank vanishes; (*b*) the normal velocity component representations at the exterior boundary are orthogonal to each other; (*c*) any normal velocity perturbation at the cylindrical wall can be expanded as a converging series in terms of the eigenfunctions in this sequence.

(*B*) *The case* $\lambda^2 < 0$

The main purpose for constructing these eigenfunctions is that any normal velocity perturbation at the exterior boundary can be expanded as a converging series in terms of these eigenfunctions. Hence the choice of a boundary condition for the eigenfunction is more or less free. In the present case there is no straightforward boundary condition that can be prescribed such as in case (*A*). The following boundary condition has been selected

$$\left.\frac{\partial w}{\partial \eta}\right|_{\eta=\eta_0} = 0. \tag{33}$$

We choose this boundary condition because this is the weakest boundary condition among a variety of possible ones, and the eigenfunctions thus obtained do not depend on the parameter ω.

For this case the following sequence of eigenfunctions is obtained

$$\psi_n = J_m(\lambda_n \eta) \cosh(\lambda_n \mu)\, e^{i(\omega t + m\theta)}, \tag{34}$$

where J_m is the Bessel function of order m, and λ_n are the solutions, put in ascending order according to their magnitude, of the equation

$$J'_m(\lambda_n \eta_0) = \left.\frac{dJ_m(\lambda_n \eta)}{d\eta}\right|_{\eta=\eta_0} = 0. \tag{35}$$

The corresponding velocity components u and w derived from the above eigenfunctions are:

$$u_n = \frac{i\lambda_n}{\omega^2-4}\left\{\frac{\omega}{2} J'_m(\lambda_n \eta) + \frac{2m}{\lambda_n \eta} J_m(\lambda_n \eta)\right\} \cosh(\lambda_n \mu)\, e^{i(\omega t + m\theta)},$$

$$w_n = \frac{i\lambda_n \omega}{\omega^2-4} J_m(\lambda_n \eta) \sinh(\lambda_n \mu)\, e^{i(\omega t + m\theta)}.$$

The properties of this sequence of eigenfunctions are summarized as follows: (a) after orthogonalization with respect to the first sequence of the eigenfunctions as obtained in the case (A), each of the eigenfunctions obtained in this case will satisfy the homogeneous boundary condition at the cylindrical boundary. (b) in the domain restricted to the end disks of the tank, the normal velocity component representations are orthogonal to each other. Any normal velocity perturbation in this domain can be expanded as a converging series of the eigenfunctions in this sequence.

The eigenfunctions of sequences (A) and (B), evaluated at the exterior boundary and normalized, can be written as

Sequence (A)

$$\left.\begin{aligned} \phi_n &= \cos\left(\frac{n\pi}{l^*}\mu\right) \quad \text{at the cylindrical wall,} \\ \phi_n &= \frac{I_m\left(\frac{n\pi}{l^*}\eta\right)}{I_m\left(\frac{n\pi}{l^*}\eta_0\right)} \quad \text{at the plane end disks,} \end{aligned}\right\} \tag{36}$$

Sequence (B)

$$\left.\begin{aligned} \psi_n &= J_m(\lambda_n \eta_0) \frac{\cosh(\lambda_n \mu)}{\cosh(\lambda_n l^*)} \quad \text{at the cylindrical wall,} \\ \psi_n &= J_m(\lambda_n \eta) \quad \text{at the plane end disks,} \end{aligned}\right\} \tag{37}$$

with $n = 1, 2, \ldots$.

We shall form a single sequence of eigenfunctions out of the sequences (A) and (B) so that this single sequence is the final sequence we wanted.

By returning to the prolate spheroidal co-ordinates, the zeros of the function $P_n^m(\beta)$ are uniformly distributed along the φ-variable, where φ is defined as $\cos\varphi = \beta$. Let the co-ordinate surface passing the intersections of the cylinder with the end disks be denoted by $\beta = \pm\beta_0$. For sufficiently large numbers of n, the number of zeros of the function $P_n^m(\beta)$ outside the hyperboloids and the number of zeros inside approaches a constant ratio

$$\nu = \frac{\frac{1}{2}\pi - \varphi_0}{\varphi_0}, \tag{38}$$

where $\cos\varphi_0 = \beta_0$, and $\varphi_0 < \frac{1}{2}\pi$.

The new sequence of eigenfunctions $\{\Phi_n\}$ is denumerated from the sequence (A) and the sequence (B) in such a way that in the first N terms of the new sequence, the ratio of the members from sequence (A) to the members from sequence (B), shall always be the rational number closest to ν.

The eigenfunctions in the sequence $\{\Phi_n\}$ can be orthogonalized by using the Hilbert–Schmidt procedure. In the present study, only the homogeneous boundary condition has to be satisfied. The sequence $\{\Phi_n\}$ can be used directly. The results obtained are equivalent to those obtained by using the orthogonalized sequence of eigenfunctions.

In the integrals (28), the function P and the eigenfunctions Φ_n are expressed analytically in two different co-ordinate systems. Hence, these integrals can be handled conveniently only in a numerical fashion.

Returning to the interior boundary condition (26), we find that this condition for $m \geqslant 2$ will reduce to

$$8E\{\tfrac{1}{4}a^2\alpha_0\alpha_1(1-\beta^2)+p_0\} = -(J^*-J). \tag{39}$$

The term $B\eta\, e^{i(\Omega t+\theta)}$ in equation (26) belongs to the case $m = 1$. The perturbation pressure at the interface p_0, the difference of curvatures, J^*-J, and the perturbation of the bubble shape $\alpha_1(\beta,\theta)$, can all be expressed in terms of a series of the associated Legendre functions with unknown constant coefficients. The constant ω is the eigenfrequency parameter. For p_0 we find

$$p_0 = \sum_{n=m}^{\infty} [A_{mn}P_n^m(\alpha_0)+B_{mn}Q_n^m(\alpha_0)]\,P_n^m(\beta)\,e^{i(\omega t+m\theta)}. \tag{40}$$

The dynamic boundary condition (39) is evaluated at the original equilibrium boundary of the bubble, $\alpha = \alpha_0$, thus this boundary condition is, apart from the factor $e^{i(\omega t+m\theta)}$, a function of β only. After some lengthy computations, for the quantities

$$J^*-J,\quad \alpha_1(\beta),\quad \frac{d\alpha_1(\beta)}{d\beta}\quad \text{and}\quad \frac{d^2\alpha_1(\beta)}{d\beta^2}$$

there results

$$J^*-J = \frac{f_C\,e^{i(\omega t+m\theta)}}{K^3(\alpha_0^2-1)^3}\left(1+1{\cdot}725\frac{K^2-1}{K^2}\beta^2\right)\Big\{[a_0^{(1)}+a_2^{(1)}\beta^2]\,\alpha_1(\beta)$$
$$+\frac{m^2\beta^2}{(1-\beta^2)}(\alpha_0^2-1)(\alpha_0^2-\beta^2)\,\alpha_1(\beta)+[a_0^{(2)}+a_2^{(2)}\beta^2]\,\beta\frac{d\alpha_1(\beta)}{d\beta}$$

$$+(\alpha_0^2-\beta^2)(1-\beta^2)(\alpha_0^2-1)\frac{d^2\alpha_1(\beta)}{d\beta^2}\bigg\}$$
$$+\frac{f_C\,e^{i(\omega t+m\theta)}}{K^5(\alpha_0^2-1)^5}\left(1+3{\cdot}625\frac{K^2-1}{K^2}\beta^2\right)\bigg\{[a_0^{(3)}+a_2^{(3)}\beta^2+a_4^{(3)}\beta^4]\,\alpha_1(\beta)$$
$$+[a_0^{(4)}+a_2^{(4)}\beta^2+a_4^{(4)}\beta^4]\,\beta\frac{d\alpha_1(\beta)}{d\beta}\bigg\}, \tag{41}$$

$$\alpha_1(\beta)=\frac{4f_D}{a^2(\alpha^2-\beta^2)}\sum_{n=m}^{\infty}\{k_P^{(n)}(\alpha_0;m,\omega)\,A_{mn}+k_Q^{(n)}(\alpha_0;m,\omega)\,B_{mn}\}\,P_n^m(\beta), \tag{42}$$

$$\frac{d\alpha_1(\beta)}{d\beta}=\frac{4f_D}{a^2(\alpha_0^2-\beta^2)^2(1-\beta^2)}\sum_{n=m}^{\infty}\{k_P^{(n)}(\alpha_0;m,\omega)\,A_{mn}+k_Q^{(n)}(\alpha_0;m,\omega)\,B_{mn}\}$$
$$\times\{[2+(n+1)\alpha_0^2-(m+3)\beta^2]\,\beta P_n^m(\beta)-(n-m+1)(\alpha_0^2-\beta^2)\,P_{n+1}^m(\beta)\}, \tag{43}$$

$$\frac{d^2\alpha_1(\beta)}{d\beta^2}=\frac{4f_D}{a^2(\alpha_0^2-\beta^2)^3(1-\beta^2)}\sum_{n=m}^{\infty}\{k_P^{(n)}(\alpha_0;m,\omega)\,A_{mn}+k_Q^{(n)}(\alpha_0;m,\omega)\,B_{mn}\}$$
$$\times\{[a_0^{(5)}+a_2^{(5)}\beta^2+a_4^{(5)}\beta^4+a_6^{(5)}\beta^6]\,P_n^m(\beta)+[a_0^{(6)}+a_2^{(6)}\beta^2+a_4^{(6)}\beta^4]\,\beta P_n^m(\beta)\}, \tag{44}$$

where the $a_j^{(i)}$, f_C, f_D, as well as the $k_P^{(n)}$ and $k_Q^{(n)}$ are constants. They have been documented by Pao (1967).

The factor $(1-\beta^2)$ vanishes at $\beta=\pm 1$. However, the terms involving this factor in the denominator remain regular due to the fact that the associated functions $P_n^m(\beta)$ have a zero of the same or higher order at these points when $m\geqslant 2$. It is permissible to multiply the equation of the boundary condition by a factor $(\alpha_0^2-\beta^2)^2(1-\beta^2)$. By doing so, the boundary condition is reduced to a summation of terms of $P_n^m(\beta)$ with polynomials of β as the coefficients. It appears in the form

$$\sum_{n=m}^{\infty}\{[\mathscr{P}_n^{(1)}(\beta)\,P_n^m(\beta)+\mathscr{P}_n^{(2)}P_{n+1}^m(\beta)]+\frac{1}{\omega}[\mathscr{P}_n^{(3)}(\beta)\,P_n^m(\beta)+\mathscr{P}_n^{(4)}P_{n+1}^m(\beta)]\}=0, \tag{45}$$

where $\mathscr{P}_n^{(i)}(\beta)$ $(i=1,2,3,4)$, are polynomials of β. The highest-order terms in the polynomials are β^8. This boundary condition can be reduced further to a series of $P_n^m(\beta)$ with constant coefficient by means of the recurrence formula

$$\beta P_n^m(\beta)=\frac{1}{2n+1}\{(n-m+1)\,P_n^m(\beta)+(n+m)\,P_{n-1}^m(\beta)\}. \tag{46}$$

Then finally, the dynamic boundary condition can be written in the form

$$\sum_{n=m}^{\infty}\sum_{K=n-8}^{K=n+8}\{f_K^{(n)}(\alpha_0;m,\omega)\,A_{mK}+g_K^{(n)}(\alpha_0;m,\omega)\,B_{mK}\}\,P_n^m(\beta)=0, \tag{47}$$

where the $f_K^{(n)}(\alpha_0;m,\omega)$ and $g_K^{(n)}(\alpha_0;m,\omega)$ are constants determined by α_0, m, n, and containing ω linearly as an eigenparameter. The coefficients of the polynomials $\mathscr{P}_n^{(i)}(\beta)$, as well as the constants $f_K^{(n)}(\alpha_0;m,\omega)$ and $g_K^{(n)}(\alpha_0;m,\omega)$, which follow from equation (45) by repeated application of (46), are obtained numerically for given values of m, α_0 and ω. They are generated by the computer program and will not be reproduced here.

A system of linear algebraic equations in terms of A_{mn} and B_{mn} can be obtained by forming the scalar products of this boundary condition and the eigenfunctions $P_n^m(\beta)$ for $n\geqslant m$. Together with the exterior boundary condition, there are sufficient algebraic equations to determine the eigenvalues of ω, and subsequently the eigensolutions of A_{mn} and B_{mn}, which determine the mode shape of the oscillations of the bubble interface.

In conclusion, there is one remark concerning the effects of the surface tension and the rotation of the entire system. In the final form of the interior boundary condition, there is a second summation sign which sums over k for nine non-zero terms. In an elementary oscillation problem, the second summation sign is not present. Each simple mode of the harmonic oscillation is distinct. The surface tension here induces an interaction among nine simple modes of harmonic oscillations, while the rotation alone will induce an interaction among three consecutive simple modes.

If the series representation of the perturbation pressure is truncated to n terms only, the boundary conditions can be reduced to a system of $(n-m)+2$ linear homogeneous algebraic equations in terms of the $(n-m)+2$ unknown constants A_{mn} and B_{mn}. Each of the boundary conditions will render half the number of linear equations. The eigenvalues of ω can be obtained from this system.

The reduction of the boundary condition to a system of algebraic equations is given in the following.

The exterior boundary condition (28) yields

$$\sum_{K=m}^{m+n} \{g^*_{iK} A_{mK} + h^*_{iK} B_{mK}\} = 0 \quad (i = 1, 2, \ldots, \tfrac{1}{2}(n-m)+1), \tag{48}$$

where
$$g^*_{iK} = \oint_\Sigma \Phi_i(\eta, \theta, \mu)\, P^m_K(\alpha)\, P^m_K(\beta)\, d\sigma,$$
$$h^*_{iK} = \oint_\Sigma \Phi_i(\eta, \theta, \mu)\, Q^m_K(\alpha)\, P^m_K(\beta)\, d\sigma,$$

while the interior boundary condition (47) becomes

$$\sum_{K=m}^{m+n} \{f^{(i)}_K(\alpha_0; m, \omega)\, A_{mK} + g^{(i)}_K(\alpha_0; m, \omega)\, B_{mK}\} = 0 \quad (i = m, m+2, \ldots, m+n), \tag{49}$$

where $f^{(i)}_K(\alpha_0; m, \omega)$ and $g^{(i)}_K(\alpha_0; m, \omega)$ are the same as before. Furthermore, the coefficients can be written as

$$f^{(i)}_K(\alpha_0; m, \omega) = a^{*(1)}_{iK} - (1/\omega)\, b^{*(1)}_{iK},$$
$$g^{(i)}_K(\alpha_0; m, \omega) = a^{*(2)}_{iK} - (1/\omega)\, b^{*(2)}_{iK}.$$

At this point it is convenient to introduce matrix notation to handle the system of algebraic equations. All the matrices, G, H, $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, defined below, are $\{\frac{1}{2}(n-m)+1\} \times \{\frac{1}{2}(n-m)+1\}$ square matrices. The vectors $\mathbf{X}$ and $\mathbf{Y}$ are two column matrices with $\frac{1}{2}(n-m)+1$ elements. They are

$$\left.\begin{array}{llll}
G = \{g_{ij}\}, & g_{ij} = g^*_{i\beta} & B^{(1)} = \{b^{(1)}_{ij}\}, & b^{(1)}_{ij} = b^{*(1)}_{\alpha\beta}, \\
H = \{h_{ij}\}, & h_{ij} = h^*_{i\beta} & B^{(2)} = \{b^{(2)}_{ij}\}, & b^{(2)}_{ij} = b^{*(2)}_{\alpha\beta}, \\
A^{(1)} = \{a^{(1)}_{ij}\}, & a^{(1)}_{ij} = a^{*(1)}_{\alpha\beta} & \mathbf{X} = \{x_i\}, & x_i = A_{m\alpha}, \\
A^{(2)} = \{a^{(2)}_{ij}\}, & a^{(2)}_{ij} = a^{*(2)}_{\alpha\beta}, & \mathbf{Y} = \{y_i\}, & y_i = B_{m\alpha}, \\
\multicolumn{4}{l}{\alpha = m+2(i-1), \qquad (i,j = 1, 2, \ldots\ldots, \tfrac{1}{2}(n-m)+1).} \\
\multicolumn{4}{l}{\beta = m+2(j-1),}
\end{array}\right\} \tag{50}$$

Then the exterior boundary condition together with the interior boundary condition can be written in the following concise form

$$\left\{\left|\begin{array}{c:c} A^{(1)} & A^{(2)} \\ \hdashline G & H \end{array}\right| - \frac{1}{\omega}\left|\begin{array}{c:c} B^{(1)} & B^{(2)} \\ \hdashline 0 & 0 \end{array}\right|\right\} \cdot \left|\begin{array}{c} \mathbf{X} \\ \mathbf{Y} \end{array}\right| = 0. \tag{51}$$

The solution of this matrix equation yields

$$\mathbf{X} = -G^{-1}H\mathbf{Y} \tag{52}$$

and

$$\mathbf{Y} = S^{-1}\mathbf{Z}, \tag{53}$$

where the eigenvalues $1/\omega$ and the eigenvectors $\mathbf{Z}$ $(=S\mathbf{Y})$ are found from

$$[(A^{(2)} - A^{(1)}G^{-1}H)\,S^{-1} - (1/\omega)\,E]\,\mathbf{Z} = 0. \tag{54}$$

E is the unit matrix, while S^{-1} is the inverse of $S = B^{(2)} - B^{(1)}G^{-1}H$.

The shape factor K	The first eigenfrequency	The second eigenfrequency
1·100	2·296	2·409
1·200	2·401	2·558
1·400	2·516	2·780
1·600	2·620	2·977
1·800	2·754	3·235
2·000	2·941	3·605
2·200	3·168	4·007

The dimensions of the tank: $\eta_0 = 2{\cdot}000$, $l = 3{\cdot}000$.

TABLE 1. The first and second eigenfrequencies for bubbles with different shape factors in a cylindrical tank with fixed dimensions.

With given data on the tank dimensions, properties of the liquid, and the speed of rotation, numerical results can be obtained easily by performing the computations according to this theory on a digital computer.

The scale of the affine transformation $\omega/(\omega^2-4)^{\frac{1}{2}}$ has to be chosen at the beginning. Through the computation we obtain a new value for ω. Using these two values of ω a more accurate value of ω can be predicted. Very accurate results are obtained within two or three steps.

Numerical computations have been made for the case $m = 2$. For any specific example we are able to compute the first and the second eigenfrequencies accurately. The result shows that the first eigenfrequency above 2·000 corresponds always to the first mode in the meridian plane. Some of the numerical results are presented in table 1, figure 1 and figure 2. It is interesting to note that the excitation of the interface wave is much more pronounced in the neighbourhood of the equator than in the pole regions of the bubble.

In the numerical computations n has been taken as large as $m+12$, such that up to seven non-zero terms in the series are included. At least five non-zero terms shall be taken in order to obtain an accurate result.

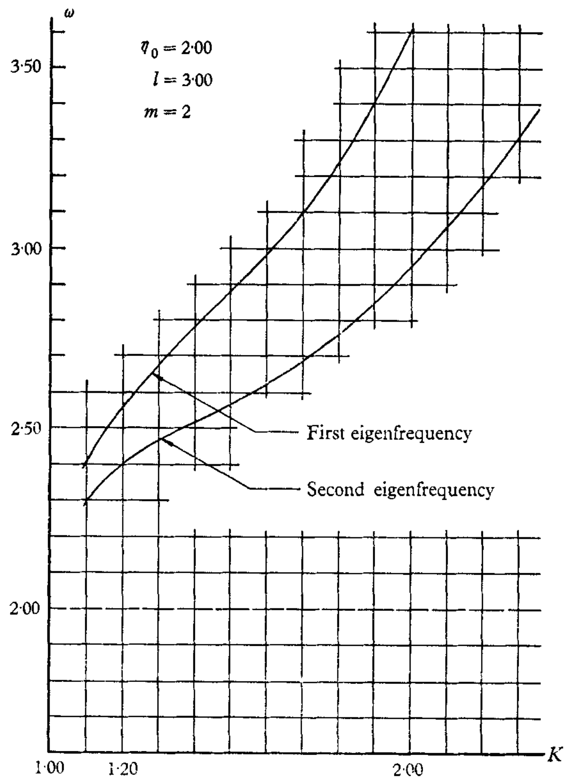

FIGURE 1. The first and second eigenfrequencies plotted versus bubble shape factor K. The bubble shape factor K is the ratio of the bubble semi-major axis to the semi-minor axis.

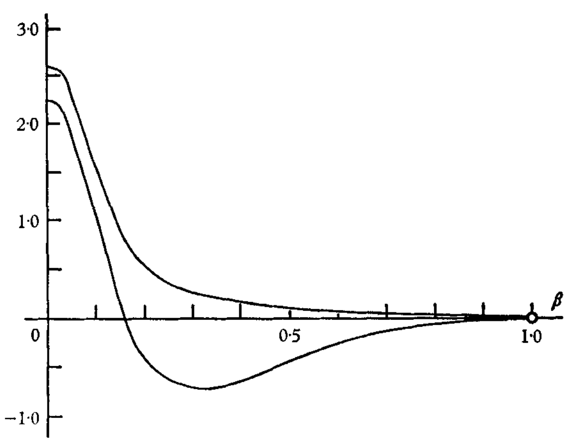

FIGURE 2. The typical normalized mode shapes in the meridian plane corresponding to the first and the second eigenfrequency of the oscillations of the bubble about its position of stable equilibrium.

4. Solution for the hyperbolic case, $\omega < 2$

In this section, the dynamical response of the rotating fluid system with respect to the perturbation due to a constant reduced gravity field, transverse to the tank axis, will be studied. We know that for the mode of oscillation $m = 1$, the motion has to be accompanied by an external force field. The governing equation is hyperbolic. Hence the method of analysis and the mathematical formulation of the problem are different from the elliptic case.

For the physical conditions given above, the frequency of the perturbation ω relative to the rotating system is one. However, the analysis for this particular case, as pursued in the following, is valid also for any frequency in the range $0 < \omega < 2$.

The mathematical formulation here is written in cylindrical co-ordinates and the 'characteristic co-ordinates'. The latter will be defined later.

Now the side conditions for a well-posed problem in the sense of Hadamard are different for hyperbolic and elliptic equations. For the present problem, we have to prescribe kinematic boundary conditions at the cylindrical wall, the disk ends and the interface, as well as a dynamic condition at the interface. Such a system of boundary conditions is too stringent for the solution to exist. We shall see that the boundary condition at the ends of the tank has to be relaxed. Hence we are considering a cylinder of infinite length instead of one with finite length.

Since the differential equation governing the pressure field is hyperbolic, there exist real characteristic surfaces in the flow domain. Across these surfaces the normal derivatives may be discontinuous. Hence the velocity field will suffer a finite jump at the same location. The work of Oser (1957) provides a good example.

The field equation in cylindrical co-ordinates (η, θ, ζ) for the case $m = 1$ and $\omega = 1$ is given by

$$\frac{\partial^2 P}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial P}{\partial \eta} - \frac{1}{\eta^2}P - 3\frac{\partial^2 P}{\partial \zeta^2} = 0. \tag{55}$$

The first-order derivative term in this equation can be eliminated by means of the transformation

$$P(\eta, \theta, \zeta) = \eta^{-\frac{1}{2}}\Phi(\eta, \theta, \zeta). \tag{56}$$

For the above hyperbolic equation, there are two families of real characteristic curves through each point of the (η, ζ)-plane. Both of these families are straight lines.

These characteristics may be chosen as our co-ordinate system. The characteristic transformation in question is

$$\left.\begin{aligned} \sqrt{3}\,\eta + \zeta &= 2\sqrt{3}\,\chi, \\ \sqrt{3}\,\eta - \zeta &= 2\sqrt{3}\,\tau, \end{aligned}\right\} \tag{57}$$

where $\chi = \text{const.}$ and $\tau = \text{const.}$ are the characteristic lines from each of the two families.

Along the characteristic directions, the transformation does not determine the second derivatives uniquely. Hence across a characteristic line, the normal derivative may suffer a jump. However, the function Φ is supposed to be continuous for a hydrodynamical problem.

Through the transformations (56) and (57), the hyperbolic differential equation (55) turns into the normal form

$$\frac{\partial^2\Phi}{\partial\chi\,\partial\tau} = \frac{3}{4}\frac{\Phi}{(\chi+\tau)^2}. \tag{58}$$

For this equation two types of problems can be posed.

(A) The Cauchy problem

The initial conditions are prescribed on a curve PQ which is nowhere tangent to the characteristic directions. Thus the values of the function Φ and its normal derivative on this curve are given. A solution exists and is uniquely determined in

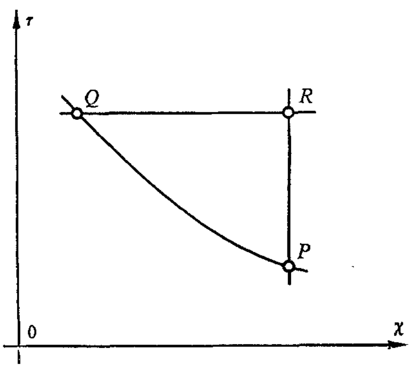

FIGURE 3. Sketch of the Cauchy problem.

a triangular domain PQR (figure 3). The entire system of differential equations and the initial conditions can be combined into the following integro-differential equation (Garabedian 1964):

$$\Phi(R) = \frac{\Phi(P)+\Phi(Q)}{2} + \frac{1}{2}\int_P^Q \left(\frac{\partial\Phi}{\partial\chi}d\chi - \frac{\partial\Phi}{\partial\tau}d\tau\right) + \iint_{PQR} \frac{3\Phi}{4(\chi+\tau)^2}\,d\chi\,d\tau. \tag{59}$$

This equation can be solved by an iteration technique. This process is proved to be convergent. The solution exists in the large when the equation is linear, as is the present case.

(B) The Goursat problem

We may also prescribe one boundary condition on the ordinary (non-characteristic) curve and one on a characteristic line. If the value of Φ is prescribed on these curves, the problem is called the Goursat problem. A solution exists also and is uniquely determined in the triangular domain OTQ (figure 4). This problem can be formulated as an integral equation (Garabedian 1964), namely,

$$\Phi(R) = \Phi(P) + \Phi(Q) - \Phi(S) + \iint_{PSQR} \frac{3\Phi}{4(\tau+\chi)^2}\,d\chi\,d\tau. \tag{60}$$

This equation can be solved by means of Picard's method of successive approximations as in (A). When the boundary condition on the ordinary curve is replaced by a mixed type condition

$$\frac{\partial\Phi}{\partial\tau} + \mathring{a}(\tau)\frac{\partial\Phi}{\partial\chi} + \mathring{b}(\tau)\,\Phi = 0, \tag{61}$$

the integro-differential equation is somewhat complicated. Let us denote the ordinary curve by

$$\chi = \chi(\tau).$$

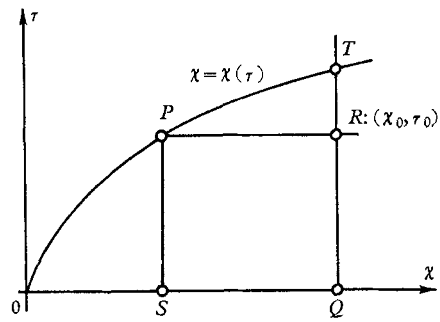

FIGURE 4. Sketch of the Goursat problem.

The function $\chi(\tau)$ is strictly monotonic. Then the integro-differential equation in question can be written in the form:

$$\Phi(R) = \Phi(Q) - \Phi(S) - \exp[-I(\tau_0)] \int_0^{T} \left[\mathring{b}(\tau)\, \Phi(\chi) + \mathring{a}(\tau) \frac{\partial \Phi}{\partial \chi} \right]_{\chi = \chi(\tau)} \exp[I(\tau)]\, d\tau + \iint_{PSQR} \frac{3\Phi}{4(\chi+\tau)^2}\, d\chi\, d\tau, \tag{62}$$

$$I(\tau) = \int_0^{\tau} \mathring{b}(\bar{\tau})\, d\bar{\tau}.$$

The solution of the Goursat problem for a linear equation exists as in case (A).

Numerical computations have been performed according to the above given integro-differential equations.

The boundary conditions prescribed for this problem have been derived in §2. They are given by equations (12), (13), (21) and (26). They are listed below as rewritten in the cylindrical co-ordinates (η, θ, ζ).

The kinematic boundary conditions at the walls of the tank

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial \eta} + \frac{1 \cdot 5}{\eta} \Phi &= 0 \quad \text{at the cylindrical wall,} \\ \frac{\partial \Phi}{\partial \zeta} &= 0 \quad \text{at the plane disk ends.} \end{aligned}\right\} \tag{63}$$

The kinematic boundary condition at the interface

$$v_n - i\delta = 0. \tag{64}$$

The normal velocity v_n and the normal displacement δ of the liquid–vapour interface are actually 90° out of phase when θ and t are taken into account.

The dynamic boundary condition at the interface

$$8E(\eta\delta_\eta + p_0) + B\eta = -(J^* - J),$$

where δ_η is the component of δ in the η-direction. This boundary condition, especially its right-hand side, is very lengthy. We shall replace it by two asymptotic representations. This approximation produces some quantitative errors while it demonstrates, however, the qualitative nature of the problem much more clearly.

For a small displacement δ of the bubble interface, the change of curvature may be approximated by $J^* - J = d^2\delta/ds^2$, where ds is a length element measured along the generator of the bubble interface. In the neighbourhood of the equator of the bubble, $d\zeta \simeq ds$. Hence we have the following asymptotic representation of the boundary condition

$$8E(\eta\delta_\eta + p_0) + B\eta = -\frac{d^2\delta}{d\zeta^2}. \tag{65}$$

In the neighbourhood of the poles of the bubble the curvature of the bubble is pronounced. The geometry in this region is best described through the spheroidal co-ordinates defined in the previous chapters.

Near the pole the value of φ is small. Thus the change of curvature may be written as

$$J^* - J = \frac{d^2\delta}{d\varphi^2}\left(\frac{d\varphi}{ds}\right)^2.$$

The η-component of the displacement δ, δ_η, is also very small in this region. It can be neglected. Hence we have another asymptotic representation for the dynamic boundary condition:

$$\left.\begin{aligned} 8Ep_0 + B\sin\varphi &= -\frac{d^2\delta}{d\varphi^2}\left(\frac{d\varphi}{ds}\right)^2, \\ \frac{d\varphi}{ds} &= \frac{1}{(1+(K^2-1)\sin^2\varphi)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{66}$$

The quantity $(d\varphi/ds)$ approaches one as φ approaches zero.

The combination of the kinematic condition (64) and the dynamical condition (65) at the interface provides us only with one boundary condition for the field quation (58).

The condition of symmetry at the equatorial plane

We assume that the flow field is symmetric with respect to the equatorial plane $\zeta = 0$. Hence we have the condition

$$\left.\frac{\partial\Phi}{\partial\zeta}\right|_{\zeta=0} = 0.$$

There are two more conditions for the determination of the flow field. These conditions will be introduced below in connexion with the construction of the solution to this problem.

For a hyperbolic equation discontinuities of the normal derivatives are admissible across a characteristic line. We shall construct a solution to this problem with all the possible discontinuities in mind. Figure 5 shows the division of the flow field into regions separated by characteristics where discontinuities may occur. In this figure the point G is located such that the bubble is tangent to the characteristic line.

As indicated earlier, we have to postulate two additional conditions to construct the flow field.

(*a*) In the immediate neighbourhood of the equatorial plane, the flow field is symmetric. The perturbation velocity normal to this plane vanishes. The bubble

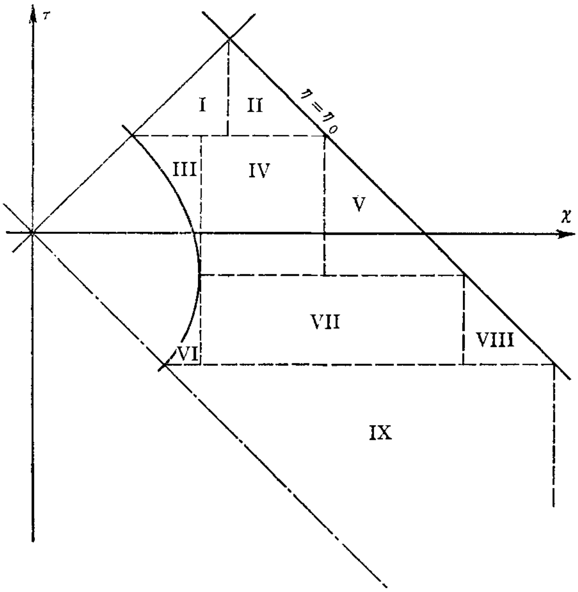

FIGURE 5. Division of the flow field into regions separated by characteristic lines where discontinuities may occur.

is tangent to a cylinder. Hence the perturbation pressure distribution given in Phillips (1960) is exactly the same as the perturbation pressure distribution at the equatorial plane. In our notation there follows

$$\Phi\Big|_{\zeta=0} = -\frac{1}{2}\frac{B}{8E}\eta^{\frac{3}{2}}\left(\frac{3}{\eta^2}-\frac{1}{\eta_0^2}\right). \tag{67}$$

(*b*) The perturbation velocity at the axis of rotation is finite. Thus we arrive at the condition

$$\Phi = 0.$$

We can now proceed with the construction of the solution. The condition of symmetry at the equatorial plane together with condition of pressure distribution, (67), form a set of initial conditions for the determination of the flow in region I. Consequently, the generalized Goursat problems can be defined in regions II to V successively. Their solutions are obtained by means of equation (60) or equation (62). In region VI, the solution is obtained as follows.

The displacement of the bubble δ at $\varphi = 0$ vanishes for the geometrical compatibility requirement. In a neighbourhood of the pole such that $\eta < |B/8E|$, both the inertia and the centrifugal field forces are smaller than the surface tension and the reduced forces by at least two orders of magnitude. Integration of equation (66) yields

$$\delta = B[(1+\tfrac{2}{3}(K^2-1)\sin\varphi+\tfrac{1}{3}\sin^3\varphi]+B\gamma\varphi, \tag{68}$$

where the constant γ is determined by the condition that the displacement at G is continuous. By means of equations (64) and (66), the solution in region VI is determined.

When we proceed to construct the solution in further regions, we find that the problem is overdetermined when the boundary condition at the disk end of the tank is prescribed. Consequently a solution to the problem would not exist in the entire flow domain at all. This boundary has to be removed for a solution to exist. Once this boundary is removed, we see immediately that the solution extends to infinity. Physically it means that a steady-state solution does not exist for a tank of finite length. The solution exists for a cylindrical tank of infinite length. In this case the perturbation propagates to infinity immediately. Such an unexpected phenomena was found in a similar problem investigated by Benjamin & Barnard (1964).

The solution in regions VII, VIII, IX, ..., can easily be determined by a sequence of generalized Goursat problems. The perturbation field approaches zero as we extend the solution to infinity along the ζ-direction.

A numerical example is given in figures 6, 7 and 8. From the results obtained from this section, we have the following physical picture for the hyperbolic

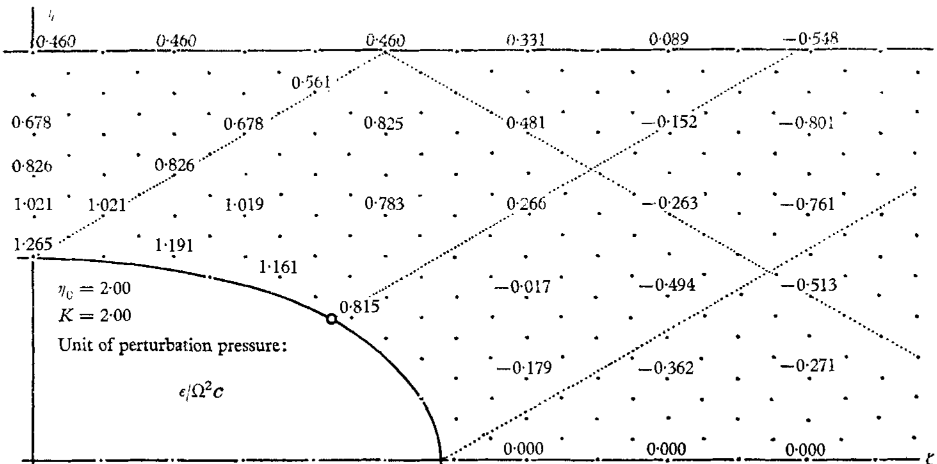

FIGURE 6. The perturbation pressure field for a tank–liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

case. The liquid–vapour system is stable with respect to the forced perturbation, since its dynamic response to the perturbation is an oscillation about the stationary equilibrium configuration with a small amplitude.

In the region $RSGA$ (figure 7), there is a strong exchange between the kinetic energy, the energy of the perturbation pressure field, and the centrifugal field effects. The effect of surface tension is negligible in this region. There is a strong secondary circulation induced by the disturbing forces.

The disturbances in the region below AG are small. Centrifugal and inertia effects are negligible in this region. The surface tension provides the necessary adjustment to absorb the pressure field perturbations. In this region, the liquid particle performs mainly an oscillatory motion in the neighbourhood of its own equilibrium position.

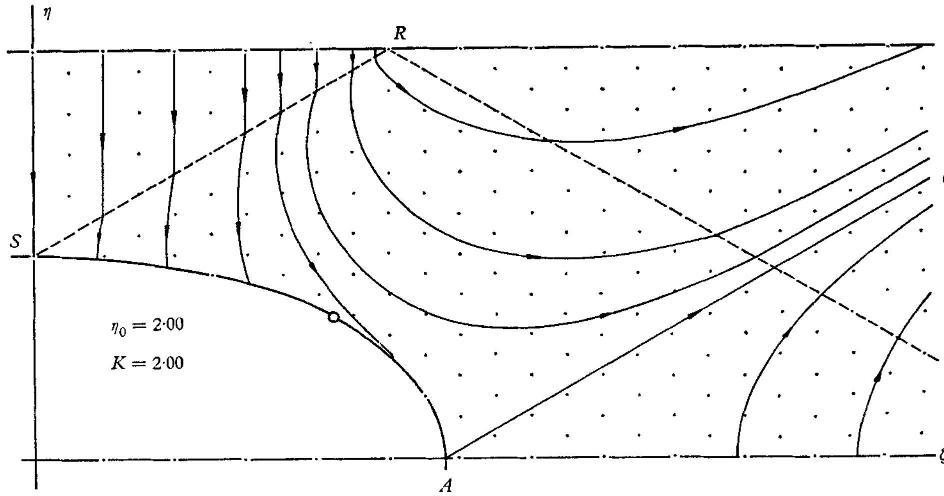

FIGURE 7. The perturbation velocity field for a tank–liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

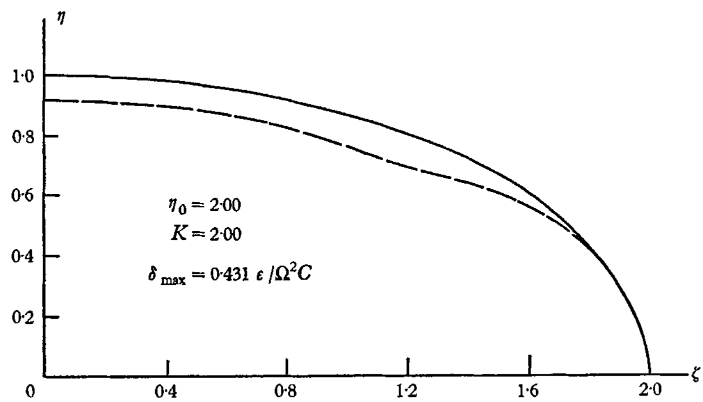

FIGURE 8. The deformed bubble shape for a tank–liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

5. Conclusions

In the previous sections we have determined the dynamic response of the rotating fluid system for the entire range of frequencies. In conclusion, we shall discuss here some of the physical implications of the results obtained above.

The slow rotation of the system with a constant angular speed has a profound effect on the dynamic response. For a rotating system, all the small oscillations of the liquid–vapour interface, or rather, of the entire liquid body, are stable. A small transverse disturbance to the system will induce one or several models of oscillation about the stable equilibrium configuration.

On the other hand, under the influence of a transverse constant force field, the stability of the configuration of the system in the neighbourhood of the equator of the bubble is ensured by the centrifugal pressure field and the inertia force produced by a small perturbation to the constant rotating base flow. The surface tension effect is negligible in this region. For a system without rotation, a disturbance containing such a force component is liable to excite instability.

Furthermore for a real system viscosity effects are always present. For a system with rotation, a perturbation will induce some secondary circulation. The disturbance will be dissipated by means of the viscous mechanism. The surface tension effect alone is a two-dimensional mechanism. It can restore a disturbed system to an equilibrium configuration in a much longer period of time.

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